TWO-WEIGHTED INEQUALITIES FOR HARDY-LITTLEWOOD MAXIMAL FUNCTIONS AND SINGULAR INTEGRALS IN $L^{p(\cdot)}$ SPACES

Vakhtang Kokilashvili and Alexander Meskhi

Abstract. Two–weight criteria of various type for the Hardy–Littlewood maximal operator and singular integrals in variable exponent Lebesgue spaces defined on the real line are established.

2010 Mathematics Subject Classification. 42B20, 42B25, 46E30.

Key words and phrases: Hardy–Littlewood maximal functions, Hilbert transforms, variable exponent Lebesgue space, weights, two–weight inequality.

Introduction

We study the two-weight problem for for Hardy–Littlewood maximal functions and singular integrals in variable exponent Lebesgue spaces $L^{p(\cdot)}$. In particular, we derive various type two–weight criteria for the maximal functions and the Hilbert transforms on the line. For a bounded interval we assume that the exponent p satisfies the local log-Hölder continuity condition and for the real line we require that p is constant outside some interval. In the framework of variable exponent analysis such a condition first appeared in the paper [4], where the author established the boundedness of the Hardy–Littlewood maximal operator in $L^{p(\cdot)}(\mathbb{R}^n)$. Unfortunately we do not know whether the established criteria remain valid or not when p satisfies log–Hölder decay condition at infinity (see [3] for this condition). It is known that the local log-Hölder continuity condition for the exponent p together with the log-Hölder decay condition guarantees the boundedness of operators of harmonic analysis in $L^{p(\cdot)}(\mathbb{R}^n)$ spaces (see [3], [26], [1], [2]).

The boundedness of the maximal, potential and singular operators in $L^{p(\cdot)}(\mathbb{R}^n)$ spaces was derived in the papers [4], [5], [7], [3], [26], [2], [1]. Weighted inequalities for classical operators in $L^{p(\cdot)}_w$ spaces, were w is a power–type weight, were established in the papers [18]-[21], [30], [27], [8] etc, while the same problems with general weights for Hardy, maximal and fractional integral operators were studied in [10]-[12], [16], [20], [22], [24], [6]. Moreover, in [6] a complete solution of the one–weight problem for maximal functions defined on Euclidean spaces are given in terms of Muckenhoupt–type conditions. Finally we notice that in the paper [12] modular–type sufficient conditions governing the two–weight inequality for maximal and singular operators were established.

Throughout the paper J denotes an interval (bounded or unbounded) in $\mathbb{R}$.

Let p be a non–negative function on $\mathbb{R}$. Suppose that E is a measurable subset of $\mathbb{R}$. We use the following notation:

$$p_-(E) := \inf_E p; \quad p_+(E) := \sup_E p; \quad p_- := p_-(\mathbb{R}); \quad p_+ := p_+(\mathbb{R}).$$

Assume that $1 \le p_-(J) \le p_+(J) < \infty$. The variable exponent Lebesgue space $L^{p(\cdot)}(J)$ (sometimes it is denoted by $L^{p(x)}(J)$) is the class of all μ-measurable functions f on X for which $S_p(f) := \int\limits_J |f(x)|^{p(x)}dx < \infty$. The norm in $L^{p(\cdot)}(J)$ is defined as follows:

$$\|f\|_{L^{p(\cdot)}(J)} = \inf\{\lambda > 0 : S_p(f/\lambda) \le 1\}.$$

It is known (see e.g. [23], [28], [18]) that $L^{p(\cdot)}$ is a Banach space. For other properties of $L^{p(\cdot)}$ spaces we refer, e.g., to [33], [23], [28].

Finally we point out that constants (often different constants in the same series of inequalities) will generally be denoted by c or C. The symbol $f(x) \approx g(x)$ means that there are positive constants c_1 and c_2 independent of x such that the inequality $f(x) \le c_1 g(x) \le c_2 f(x)$ holds. Throughout the paper by the symbol $p'(x)$ is denoted the function $p(x)/(p(x)-1)$.

1 Sawyer-type Condition for Maximal Operators in $L^{p(x)}$ Spaces.

1.1 The case of bounded interval

Let J be bounded interval in $\mathbb{R}$ and let

$$(M_\alpha^{(J)} f)(x) = \sup_{\substack{I \ni x \\ I \subset J}} \frac{1}{|I|^{1-\alpha}} \int\limits_I |f(y)|dy, \quad x \in J,$$

where $x \in J$ and α is a constant satisfying the condition $0 \le \alpha < 1$.

For a weight function u we denote

$$u(E) := \int\limits_E u(x)dx.$$

Definition 1.1. Let J be a bounded interval in $\mathbb{R}$. We say that a non–negative function u satisfies the doubling condition on J ($u \in DC(J)$) if there is a positive constant b such that for all $x \in J$ and all r, $0 < r < |J|$, the inequality

$$u\big(I(x-2r, x+2r) \cap J\big) \le b u\big(I(x-r, x+r) \cap J\big)$$

holds.

Definition 1.2. We say that $p \in LH(J)$ (p satisfies the local log-Hölder condition) if there is a positive constant c such that

$$|p(x) - p(y)| \le \frac{c}{-|x-y|}$$

for all $x, y \in J$ satisfying the condition $|x - y| \le 1/2$.

Theorem 1.1. *Let $1 < p_- \le p(x) \le p_+ < \infty$ and let the measure $d\nu(x) = w(x)^{-p'(x)}dx$ belongs to $DC(J)$. Suppose that $0 \le \alpha < 1$ and that $p \in LH(J)$. Then the inequality*

$$\|v(\cdot) M_\alpha^{(J)} f\|_{L^{p(\cdot)}(J)} \le c\|w(\cdot)f(\cdot)\|_{L^{p(\cdot)}(J)}$$

holds, if and only if there exist a positive constant c such that for all interval I, $I \subset J$,

$$\int\limits_I (v(x))^{p(x)} (M_\alpha^{(J)}(w(\cdot)^{-p'(\cdot)}\chi_{I(\cdot)}))^{p(x)}dx \le c \int\limits_I w^{-p'(x)}dx < \infty.$$

To prove Theorem 1.1 we need some auxiliary statements.

Proposition A. ([32], Lemma 3.20) *Let s be a constant satisfying the condition $1 < s < \infty$ and let $u \geq 0$ on $\mathbb{R}$. Suppose that $\{Q_i\}_{i \in A}$ is a countable collection of dyadic intervals in $\mathbb{R}$ and that $\{a_i\}_{i\in A}$,$\{b_i\}_{i\in A}$ are sequences of positive numbers satisfying the conditions:*

(i) $\int\limits_{Q_i} u \leq a_i$ *for all* $i \in A$*;*

(ii) $\sum\limits_{\{j\in A: Q_j \subset Q_i\}} b_j \leq c a_i$ *for all* $i \in A$.

Then there is a positive constant c_s depended on s such that the inequality

$$\bigg(\sum_{i\in A} b_i \bigg(\frac{1}{a_i} \int\limits_{Q_i} gu \bigg)^s \bigg)^{1/s} \leq c_s \bigg(\int\limits_{\mathbb{R}} g^s u \bigg)^{1/s}$$

holds for all non-negative functions g.

Corollary A. *Let $1 < s < \infty$ and let u be a non-negative measurable function on $\mathbb{R}$. Suppose that $\{Q_i\}_{i\in A}$ is a a sequence of dyadic cubes in $\mathbb{R}^n$ and that $\{b_i\}_{i\in A}$ is a sequence of positive numbers satisfying the condition*

$$\sum_{\{j\in A: Q_j \subset Q_i\}} b_j \leq c u(Q_i).$$

Then there is a positive constant c such that for all non-negative functions g the inequality

$$\sum_{i\in A} b_i \bigg(\frac{1}{u(Q_i)} \int\limits_{Q_i} gu \bigg)^s \leq c \bigg(\int\limits_{\mathbb{R}} g^s u \bigg)^{1/s}$$

holds.

Lemma A. *Let J be a bounded interval and let $1 \leq r_-(J) \leq r_+(J) < \infty$. Suppose that $r \in LH(J)$ and that the measure μ satisfies the condition $\mu \in DC(J)$. Then there is a positive constant c such that for all f, $\|f\|_{L^{r(\cdot)}(J,\mu)} \leq 1$, intervals $I \subseteq J$ and $x \in I$ the inequality*

$$\bigg(\frac{1}{\mu(I)} \int\limits_I |f(y)| d\mu(y) \bigg)^{r(x)} \leq c \bigg[\bigg(\frac{1}{\mu(I)} \int\limits_I |f(y)|^{r(y)} d\mu(y) \bigg) + 1 \bigg]$$

holds.

Proof. We follow the idea of L. Diening [4] (see also [14] for the similar statement in the case of metric measure spaces with doubling measure). We give the proof for completeness.

First recall that (see, e.g., [14]) since J with the Euclidean distance and the measure μ is a bounded doubling space with the finite measure μ the condition $r \in LH(J)$ implies the following inequality:

$$\big(\mu(I)\big)^{r_-(I) - r_+(I)} \leq C \tag{1.1}$$

for all subintervals I of J.

Assume that $\nu B \leq 1/2$. By Hölder's inequality we have that

$$\bigg(\frac{1}{\mu(I)} \int\limits_I |f(y)| d\mu(y) \bigg)^{r(x)} \leq \bigg(\frac{1}{\mu(I)} \int\limits_I |f(y)|^{r_-(I)} d\mu(y) \bigg)^{r(x)/r_-(I)}$$

$$\leq c\mu(I)^{-r(x)/r_-(I)}\left[\frac{1}{2}\int_I |f(y)|^{r(y)}d\mu(x)+\frac{1}{2}\mu(I)\right]^{r(x)/r_-(I)}.$$

Observe now that the expression in brackets is less than or equal to 1. Consequently, by (1.1) we find that

$$\left(\frac{1}{\mu(I)}\int_I |f(y)|d\mu(y)\right)^{r(x)} \leq c\mu(I)^{1-r(x)/r_-(I)}\left(\frac{1}{\mu(I)}\int_I |f(y)|^{r(y)}d\mu(y)+1\right)$$

$$\leq c\mu(I)^{(r_-(I)-r_+(I))/r_-(I)}\left(\frac{1}{\mu(I)}\int_I |f(y)|^{r(y)}d\mu(y)+1\right) \leq c\left(\frac{1}{\mu(I)}\int_I |f(y)|^{r(y)}d\mu(y)+1\right).$$

The case $\mu(I) > 1/2$ is trivial. $\Box$

Suppose that S is an interval in $\mathbb{R}$ and let us introduce the dyadic maximal operator

$$\big(M_\alpha^{(d),S}\big)f(x) = \sup_{\substack{x\in I\\ I\in D(S)}} |I|^{\alpha-1}\int_I |f(y)|dy,$$

where $0\leq\alpha<1$ and $D(S)$ is a dyadic lattice in S.

To prove Theorem 1.1 we need the following statement:

Lemma 1.1. *Let S be a bounded interval on $\mathbb{R}$ and let J be a subinterval of S. Suppose that $\sigma(x) := w^{-p'(x)}$ belongs to the class $DC(J)$ and that $p\in LH(J)$, where $1<p_-(J)\leq p(x)\leq p_+(J)<\infty$. Let $0\leq\alpha<1$. If there is a positive constant c such that for all interval I, $I\subset J$,*

$$\int_I (v(x))^{p(x)}\Big(M_\alpha^{(d),S}\big(\chi_I(\cdot)\sigma(\cdot)\big)\Big)^{p(x)}(x)dx \leq c\int_I \sigma(x)dx<\infty,$$

then the estimate

$$\|v(\cdot)M_\alpha^{(d),S}\big(f(\cdot)\chi_J(\cdot)\big)\|_{L^{p(\cdot)}(J)} \leq c\|w(\cdot)f(\cdot)\|_{L^{p(\cdot)}(J)}$$

holds.

Proof. Suppose that $\|f\|_{L_w^{p(\cdot)}(J)}\leq 1$. Assume that $f_1 := \chi_J f$. Let us introduce the set

$$J_k = \{x\in S : 2^k < (M_\alpha^{(d),S}f_1)(x)\leq 2^{k+1}\},\ \ k\in\mathbb{Z}.$$

Suppose that for k, $J_k\neq\emptyset$, $\{I_j^k\}$ is a maximal dyadic interval, $I_j^k\subset D(S)$, such that

$$\frac{1}{|I_j^k|^{1-\alpha}}\int_{I_j^k}|f_1(y)|dy > 2^k. \tag{1.2}$$

It is obvious that such a maximal interval always exists. Now observe that

(*i*) $\{I_j^k\}$ are disjoint for fixed k;

(*ii*)

$$\overline{J}_k := \{x\in S : \big(M_\alpha^{(d),S}f_1\big)(x) > 2^k\} = \cup_j I_j^k.$$

Indeed, (i) holds because if $I_i^k\cap I_j^k\neq\emptyset$, then $I_i^k\subset I_j^k$ or $I_j^k\subset I_i^k$. Consequently, if $I_i^k\subset I_j^k$, then I_j^k is maximal interval for which (1.2) holds.

To see that (ii) holds, observe that if $x \in \overline{J}_k$, then $M_\alpha^{(d),S} f_1(x) \geq 2^k$. Hence, there is a maximal dyadic interval I_j^k containing x such that (1.2) hold for I_j^k. Let now $x \in \bigcup\limits_j I_j^k$. Then $x \in I_{j_0}^k$ for some j_0. Hence, $M_\alpha^{(d),S} f_1(x) > 2^k$ because (1.2) holds for $I_{j_0}^k$.

Denote:

$$E_j^k := I_j^k \backslash \{x \in S : M_\alpha^{(d),S} f_1(x) > 2^{k+1}\}.$$

Then $E_j^k = I_j^k \cap J_k$. Indeed, if $x \in E_j^k$, then $x \in I_j^k$ and $M_\alpha^{(d),S} f_1(x) \leq 2^{k+1}$. Hence, by (1.2) we find that

$$2^k < |I_i^k|^{\alpha-1} \int\limits_{I_j^k} |f_1(y)| dy \leq M_\alpha^{(d),S} f_1(x) \leq 2^{k+1}.$$

This means that $x \in I_j^k \cap J_k$. Let now $x \in I_j^k \cap J_k$. Then obviously $M_\alpha^{(d),S} f_1(x) \leq 2^{k+1}$. Consequently, $x \in E_j^k$.

Observe that $\{E_j^k\}$ are disjoint for every j, k because, as we have seen,

$$E_j^k = \{x \in I_j^k : 2^k < M_\alpha^{(d),S} f_1(x) \leq 2^{k+1}\}.$$

Also, $E_j^k \subset I_j^k$. Assume that $\|w(\cdot) f_1(\cdot)\|_{L^{p(\cdot)}(S)} \leq 1$. Denote:

$$v_1 := v\chi_J, \quad \sigma_1 := \sigma\chi_J.$$

By the arguments observed above and using Lemma A with $r(\cdot) = p(\cdot)/p_-$ and the measure

$d\mu(x) = \sigma(x)dx$ we have that

$$
\begin{aligned}
& \int\limits_J (v(x))^{p(x)} \Big(M_\alpha^{(d),S} f_1 \Big)^{p(x)} (x) dx \\
= \ & \int\limits_S (v_1(x))^{p(x)} \Big(M_\alpha^{(d),S} f_1 \Big)^{p(x)} (x) dx \\
\leq \ & \sum_{j,k} \int\limits_{E_j^k} (v_1(x))^{p(x)} 2^{(k+1)p(x)} dx \\
\leq \ & c \sum_{j,k} \int\limits_{E_j^k} (v_1(x))^{p(x)} \Bigg(\frac{1}{|I_j^k|^{1-\alpha}} \int\limits_{I_j^k} |f_1(y)| dy \Bigg)^{p(x)} dx \\
= \ & c \sum_{j,k} \int\limits_{E_j^k} (v_1(x))^{p(x)} \Bigg(\frac{\sigma(I_j^k \cap J)}{|I_j^k|^{1-\alpha}} \Bigg)^{p(x)} \Bigg(\frac{1}{\sigma(I_j^k \cap J)} \int\limits_{I_j^k} \Big| \frac{f_1}{\sigma} \Big| \sigma \Bigg)^{p(x)} dx \\
= \ & c \sum_{j,k} \int\limits_{E_j^k} (v_1(x))^{p(x)} \Bigg(\frac{\sigma(I_j^k \cap J)}{|I_j^k|^{1-\alpha}} \Bigg)^{p(x)} \Bigg(\frac{1}{\sigma(I_j^k \cap J)} \int\limits_{I_j^k} \Big| \frac{f_1}{\sigma} \Big| \sigma \Bigg)^{p(x)} dx \\
\leq \ & c \sum_{j,k} \Bigg(\int\limits_{E_j^k} (v_1(x))^{p(x)} \Bigg(\frac{\sigma(I_j^k \cap J)}{|I_j^k|^{1-\alpha}} \Bigg)^{p(x)} dx \Bigg) \Bigg(\frac{1}{\sigma(I_j^k \cap J)} \int\limits_{I_j^k} \Big| \frac{f_1(y)}{\sigma(y)} \Big|^{\frac{p(y)}{p_-}} \sigma(y) dy \Bigg)^{p_-} \\
+ \ & c \sum_{j,k} \Bigg(\int\limits_{E_j^k} (v_1(x))^{p(x)} \Bigg(\frac{\sigma(I_j^k \cap J)}{|I_j^k|^{1-\alpha}} \Bigg)^{p(x)} dx \Bigg) \\
\equiv \ & c \Big(\sum_{j,k} A_j^k + \sum_{j,k} B_j^k \Big).
\end{aligned}
$$

Notice that the sign of sum is taken over all those j ad k for which $\sigma(I_j^k \cap J) > 0$).

To use Corollary A observe that

$$
\begin{aligned}
& \sum_{\substack{I_j^k \subset I_i \\ I_j^k, I_i \in D(S)}} \int\limits_{E_j^k} (v_1(x))^{p(x)} \Bigg(\frac{\sigma(I_j^k \cap J)}{|I_j^k|^{1-\alpha}} \Bigg)^{p(x)} dx \\
\leq \ & \sum_{I_j^k \subset I_i} \int\limits_{E_j^k} (v_1(x))^{p(x)} \Big(M_\alpha^{(d),S} (\chi_{I_i \cap J} \sigma) \Big)^{p(x)} (x) dx \\
\leq \ & \int\limits_{I_i} (v_1(x))^{p(x)} \Big(M_\alpha^{(d),S} (\chi_{I_i \cap J} \sigma) \Big)^{p(x)} (x) dx \\
\leq \ & c \int\limits_{I_i \cap J} \sigma(x) dx = c \int\limits_{I_i} \sigma_1(x) dx.
\end{aligned}
$$

Now Corollary A implies that

$$\sum_{j,k} A_j^k = \sum_{j,k} \Bigg(\int_{E_j^k} (v_1(x))^{p(x)} \bigg(\frac{\sigma(I_j^k \cap J)}{|I_j^k|^{1-\alpha}} \bigg)^{p(x)} dx \Bigg) \Bigg(\frac{1}{\sigma_1(I_j^k)} \int_{I_j^k} \bigg| \frac{f_1(y)}{\sigma(y)} \bigg|^{\frac{p(x)}{p_-}} \sigma_1(y)dy \Bigg)^{p_-}$$

$$\leq c \int_S |f_1(x)|^{p(x)} \sigma(x)^{-p(x)} \sigma_1(x)dx = c \int_S |f_1(x)|^{p(x)} w^{p(x)} dx \leq c.$$

For the second term we have that

$$\begin{aligned} \sum_{j,k} B_j^k &= \sum_{j,k} \int_{E_j^k} (v_1(x))^{p(x)} \bigg(\frac{\sigma(I_j^k \cap J)}{|I_j^k|^{1-\alpha}} \bigg)^{p(x)} dx \\ &\leq \sum_{j,k} \int_{E_j^k} (v_1(x))^{p(x)} \Big(M_\alpha^{(d),S}(\chi_J \sigma) \Big)^{p(x)} (x)dx \\ &= \int_J (v(x))^{p(x)} \Big(M_\alpha^{(d),S}(\chi_J \sigma) \Big)^{p(x)} (x)dx \\ &\leq c \int_J \sigma(x)dx < \infty. \end{aligned}$$

Finally we conclude that

$$\|v(\cdot)\big(M_\alpha^{(d),S} f_1\big)(\cdot)\|_{L^{p(\cdot)}(J)} \leq c$$

for $\|w(\cdot)f(\cdot)\|_{L^{p(\cdot)}(J)} \leq 1$. $\square$

Proof of Theorem 1.1. *Sufficiency.* Let us take an interval S containing J. Without loss of generality we can assume that S is a maximal dyadic interval and that $|J| \leq \frac{|S|}{8}$. Further, suppose also that J and S have one and the same center. Without loss of generality assume that $|S| = 2^{m_0}$ for some integer m_0. Then every interval $I \subset J$ has the length $|I|$ less than or equal to 2^{m_0-3} . Assume that $|I| \in [2^j, 2^{j+1})$ for some j, $j \leq m_0 - 4$. Let us introduce the set

$$F = \{t \in (-2^{m_0-4}, 2^{m_0-4}) : \text{ there is } I_1 \in D(S) - t, I \subset I_1 \subset S, |I_1| = 2^{j+1}\}.$$

The simple geometric observation (see also [13], p. 431) shows that $|F| \geq 2^{m_0-4}$.

Further, let

$$(K_t f)(x) := \sup_{\substack{S \supset I_1 \ni x \\ I_1 \in D(S)-t}} \frac{1}{|I_1|^{1-\alpha}} \int_{I_1} |f_1|, \quad t \in F,$$

where $f_1 = \chi_J f$. Then for x ($x \in J$) there exist $I \ni x$, $I \subset J$ such that

$$|I|^{\alpha-1} \int_I |f_1| > \frac{1}{2} (M_\alpha^{(J)} f_1)(x).$$

For the interval I, we have that $|I| \in [2^j, 2^{j+1})$, $j \leq m_0 - 4$. Therefore for $t \in F$, there is an interval I_1, $I_1 \in D(S) - t, I \subset I_1 \subset S, |I_1| = 2^{j+1}$, such that

$$|I|^{\alpha-1} \int_I |f_1| \leq \frac{c}{|I_1|^{1-\alpha}} \int_{I_1} |f_1|.$$

Hence,

$$(M_\alpha^{(J)} f)(x) \le c(K_t f_1)(x), \ \text{ for every } \ t \in F, x \in J,$$

with the positive constant c depending only on α. Consequently,

$$\begin{aligned}(M_\alpha^{(J)} f)(x) &\le \frac{1}{|F|} \int_F (K_t f_1)(x) dt \\ &\le \frac{c}{|I(0, 2^{m_0 - 4})|} \int_{I(0,2^{m_0-4})} (K_t f_1)(x) dt.\end{aligned}$$

Suppose that $\|w(\cdot) f(\cdot)\|_{L^{p(\cdot)}(J)} \le 1$. Then by Lemma 1.1 we have that

$$\begin{aligned} S_t &:= \int_J (v(x))^{p(x)} \big((K_t f_1)(x) \big)^{p(x)} dx \\ &= \int_J (v(x))^{p(x)} \Bigg(\sup_{\substack{S \supset I_1 \ni x \\ I_1 \in D(S) - t}} \frac{1}{|I_1|} \int_{I_1} |f_1| \Bigg)^{p(x)} dx \\ &= \int_{J+t} (v_t(x))^{p(x-t)} \Bigg(\sup_{\substack{S \supset I_1 \ni x \\ I_1 \in D(S)}} |I_1|^{\alpha - 1} \int_{I_1} \chi_J(s-t) f_1(s-t) ds \Bigg)^{p(x-t)} dx \\ &= \int_{J+t} (v_t(x))^{p_t(x)} \Bigg(\sup_{\substack{I_1 \ni x \\ I_1 \in D(S)}} |I_1|^{\alpha - 1} \int_{I_1} \chi_{J+t}(s) f_1(s-t) ds \Bigg)^{p_t(x)} dx \\ &= \int_{J+t} (v_t(x))^{p_t(x)} \Big(M_\alpha^{(d),S} \big(\chi_{J+t}(\cdot) f_1(\cdot - t) \big) \Big)^{p_t(x)} dx \\ &\le c \end{aligned}$$

provided that

$$\int_{J+t} (w_t(x))^{p_t(x)} (f_1(x-t))^{p_t(x)} dx = \int_J w(x) |f(x)|^{p(x)} dx \le 1,$$

where $v_t(x) = v(x-t)$, $w_t(x) = w(x-t)$, $p_t(x) = p(x-t)$. To justify this conclusion we need to check that for every $I, I \subset J+t$,

$$\int_I (v_t(x))^{p_t(x)} \Big(M_\alpha^{(d),S} (\sigma_t \chi_I)(x) \Big)^{p_t(x)} dx \le c \int_I \sigma_t(x) dx < \infty,$$

where the positive constant c is independent of I and t. Indeed, observe that

$$\begin{aligned}
&\int_I (v_t(x))^{p_t(x)} \Big(M_\alpha^{(d),S}(\sigma_t\chi_I)(x)\Big)^{p_t(x)} dx \\
&= \int_I (v_t(x))^{p_t(x)} \Bigg(\sup_{\substack{I_1\ni x\\ I_1\in D(S)}} |I_1|^{\alpha-1} \int_{I_1} \chi_I(s)\sigma(s-t)ds\Bigg)^{p_t(x)} dx \\
&= \int_I (v_t(x))^{p_t(x)} \Bigg(\sup_{\substack{I_1-t\ni x-t\\ I_1\in D(S)}} |I_1-t|^{\alpha-1} \int_{I_1-t} \chi_I(s+t)\sigma(s)ds\Bigg)^{p_t(x)} dx \\
&= \int_{I-t} (v(x))^{p(x)} \Bigg(\sup_{\substack{I_1\ni x\\ I_1\in D(S)-t}} |I_1|^{\alpha-1} \int_{I_1} \chi_{I-t}(s)\sigma(s)ds\Bigg)^{p(x)} dx \\
&\le \int_{I-t} (v(x))^{p(x)} \Big(M_\alpha^{(J)}(\chi_{I-t}\sigma)\Big)^{p(x)}(x)dx \le \int_{I-t} \sigma(x)dx \\
&= \int_I \sigma_t(x)dx < \infty.
\end{aligned}$$

Further, let $g \in L^{p'(\cdot)}(J)$ with $\|g\|_{L^{p'(\cdot)}(J)} \le 1$. Then we find that

$$\begin{aligned}
&\int_J (M_\alpha^{(J)} f)(x)v(x)g(x)dx \\
&\le \int_J \Bigg(\frac{1}{|I(0,2^{m_0-4})|} \int_{I(0,2^{m_0-4})} (K_t f_1)(x)dt\Bigg) v(x)g(x)dx \\
&\le \frac{1}{|I(0,2^{m_0-4})|} \int_{I(0,2^{m_0-4})} \Bigg(\int_J (K_t f_1)(x)g(x)v(x)dx\Bigg) dt \\
&\le \frac{1}{|I(0,2^{m_0-4})|} \int_{I(0,2^{m_0-4})} \|(K_t f_1)v\|_{L^{p(\cdot)}(J)} \|g\|_{L^{p'(\cdot)}(J)} dt \\
&\le c,
\end{aligned}$$

provided that $\|f\|_{L^{p(\cdot)}_w(J)} \le 1$.

Finally we conclude that $\|(M_\alpha^{(J)} f)v\|_{L^{p(\cdot)}(J)} \le c$ if $\|fw\|_{L^{p'(\cdot)}(J)} \le 1$.

Sufficiency is proved.

Necessity. Let $f_I(t) = \chi_I(t)w^{-p'(t)}(t)$. Suppose that $\beta = \|w^{-1}(\cdot)\|_{L^{p'(\cdot)}(J)} \le 1$. We have that

$$\big\|v(\cdot)(M_\alpha^{(J)} f)^{p(\cdot)}(\cdot)\big\|_{L^{p(\cdot)}(J)} \ge \big\|\chi_I(\cdot)v(\cdot)\big(M_\alpha^{(J)}\big(w^{-p'(\cdot)}(\cdot)\chi_I(\cdot)\big)\big)(\cdot)\big\|_{L^{p(\cdot)}(J)} =: A.$$

Hence, by the boundedness of $M_\alpha^{(J)}$, Lemma B (recall that the measure $d\nu(x) = w(x)^{-p'(x)}dx$ satisfies the doubling condition) and the fact that $1/p \in LH(J)$ we find that

$$
\begin{aligned}
A = &\ \big\|\chi_I(\cdot)v(\cdot)M_\alpha^{(J)}\big(w^{-p'(\cdot)}(\cdot)\chi_I(\cdot)\big)(\cdot)\big\|_{L^{p(\cdot)}(J)}\\
\le &\ c\big\|w(\cdot)w^{-p'(\cdot)}(\cdot)\chi_I(\cdot)\big\|_{L^{p(\cdot)}(J)}\\
\le &\ c\Big(\int\limits_I w^{-p'(x)p(x)}(x)w^{p(x)}(x)dx\Big)^{1/p_+(I)}\\
\le &\ \bar{c}\Big(\int\limits_I w^{-p'(x)}(x)dx\Big)^{\frac{1}{p_-(I)}} \le \bar{c}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
A = &\ \bar{c}\bigg\|\frac{1}{\bar{c}}\chi_I(\cdot)v(\cdot)M_\alpha^{(J)}\big(w^{-p'(\cdot)}\chi_I(\cdot)\big)(\cdot)\bigg\|_{L^{p(\cdot)}(J)}\\
\ge &\ \bar{c}\Big(\int\limits_I (\bar{c})^{-p(x)}(v(x))^{p(x)}\Big[M_\alpha^{(J)}\big(w^{-p'(\cdot)}\chi_I(\cdot)\big)\Big](x)dx\Big)^{\frac{1}{p_-(I)}}\\
\ge &\ c\Big[\int\limits_I (v(x))^{p(x)}\Big(M_\alpha^{(J)}\big(w^{-p'(\cdot)}\chi_I(\cdot)\big)(x)\Big)^{p(x)}dx\Big]^{\frac{1}{p_-(I)}}.
\end{aligned}
$$

Summarizing these inequalities we conclude that

$$
\int\limits_I (v(x))^{p(x)}\Big(M_\alpha^{(J)}\big(w^{-p'(\cdot)}\chi_I(\cdot)\big)(x)\Big)^{p(x)}dx \le c\int\limits_I w^{-p'(x)}(x)dx < \infty.
$$

Suppose now that $\beta \ge 1$. Let us take

$$
f(t) = \frac{w^{-p'(t)}(t)\chi_I(t)}{\beta}.
$$

Then

$$
\|f_I(\cdot)w(\cdot)\|_{L^{p(\cdot)}(J)} = \frac{\|w^{1-p'(\cdot)}(\cdot)\chi_I(\cdot)\|_{L^{p(\cdot)}(J)}}{\beta} \le 1.
$$

Arguing as above we have desire result. It remains to show that

$$
A := \int\limits_J w^{-p'(x)}(x)dx < \infty.
$$

Suppose that $A = \infty$. Then $\|w^{-1}(\cdot)\|_{L^{p'(\cdot)}(J)} = \infty$. Hence, there exist a function g, $\|g\|_{L^{p(\cdot)}(J)}, g \ge 0$ such that

$$
\int\limits_J g(x)w^{-1}(x)dx = \infty.
$$

Let $f(x) = g(x)w^{-1}(x)$. Then

$$
\Big\|v(\cdot)\big(M_\alpha^{(J)}f\big)(\cdot)\Big\|_{L^{p(\cdot)}(J)} \ge \Big(\int\limits_J w^{-1}(x)g(x)\Big)\Big\|v(\cdot)|J|^{\alpha-1}\Big\|_{L^{p(\cdot)}(J)} = \infty,
$$

while

$$\|fw\|_{L^{p(\cdot)}(J)} = \|g\|_{L^{p(\cdot)}(J)} < \infty.$$

$\square$

Corollary 1.1. *Let J be a bounded interval and let $1 < p_-(J) \le p(x) \le p_+(J) < \infty$ and let $0 \le \alpha < 1$. Assume that $p \in LH(J)$ then the inequity*

$$\big\|v(\cdot)\big(M_\alpha^{(J)} f\big)(\cdot)\big\|_{L^{p(\cdot)}(J)} \le c\|f\|_{L^{p(\cdot)}(J)} \ \ (\textit{Trace inequality})$$

holds if and only if

$$\sup_{I, I\subset J} \frac{1}{|I|} \int_I (v(x))^{p(x)} |I|^{\alpha p(x)} dx < \infty.$$

Proof. Sufficiency. By Theorem 1.1 it is enough to see that

$$\big(M_\alpha^{(J)} \chi_I\big)(x) \le |I|^\alpha \quad \text{for} \quad x \in I.$$

This is true because of the following estimates:

$$\sup_{\substack{S, S\subset J\\ S\ni x}} |S|^{\alpha-1} \int_S \chi_I \le \sup_{\substack{S\cap I\ni x\\ S\subset J}} |S\cap I|^{\alpha-1} \int_{S\cap I} dx = \sup_{\substack{S\cap I\ni x\\ S\subset J}} |S\cap I|^\alpha = |I|^\alpha.$$

Necessity follows by choosing the appropriate test functions in the trace inequality. $\square$

1.2 The case of unbounded interval

Now we derive criteria for the two–weight inequality for the following maximal operators:

$$\left(M_\alpha^{(\mathbb{R}_+)} f\right)(x) = \sup_{h>0} \frac{1}{h^{1-\alpha}} \int_{(x-h, x+h)\cap\mathbb{R}_+} |f(y)| dy$$

and

$$\left(M_\alpha^{(\mathbb{R})} f\right)(x) = \sup_{h>0} \frac{1}{h^{1-\alpha}} \int_{x-h}^{x+h} |f(y)| dy,$$

where $0 \le \alpha < 1$.

In the sequel we will assume that $v^{p(\cdot)}(\cdot)$ and $w^{-p'(\cdot)}(\cdot)$ are a.e. positive locally integrable function.

Theorem 1.2. *Let $0 \le \alpha < 1$, $1 < p_-(\mathbb{R}_+) \le p \le p_+(\mathbb{R}_+) < \infty$ and let $p \in LH(\mathbb{R}_+)$. Suppose that there is a bounded interval $[0, a]$ such that $w^{-p'(\cdot)}(\cdot) \in DC([0, a])$ and $p \equiv p_c \equiv$const outside $[0, a]$. Then the inequity*

$$\|vM_\alpha^{(\mathbb{R}_+)} f\|_{L^{p(\cdot)}(\mathbb{R}_+)} \le \|wf\|_{L^{p(\cdot)}(\mathbb{R}_+)},$$

holds if and only if there is a positive constant b such that for all bounded intervals $I \subset \mathbb{R}_+$,

$$\|vM_\alpha^{(\mathbb{R}_+)}(w^{-p'(\cdot)}\chi_I)\|_{L^{p(\cdot)}(I)} \le c\|w^{1-p'(\cdot)}\|_{L^{p(\cdot)}(I)} < \infty. \tag{1.3}$$

Proof. *Sufficiency.* Suppose that $\|wf\|_{L^{p(\cdot)}(\mathbb{R}_+)} < \infty$. We will show that $\|vM_\alpha^{(\mathbb{R}_+)}\|_{L^{p(\cdot)}(\mathbb{R}_+)} < \infty$.

Represent $M_\alpha^{(\mathbb{R}_+)} f(x)$ as follows:

$$\begin{aligned} & M_\alpha^{(\mathbb{R}_+)} f(x) = \chi_{[0,a]}(x) M_\alpha^{(\mathbb{R}_+)}\big(f\cdot\chi_{[0,a]}\big)(x) \\ & +\chi_{[0,a]}(x) M_\alpha^{(\mathbb{R}_+)}\big(f\cdot\chi_{(a,\infty)}\big)(x) + \chi_{(a,\infty)}(x) M_\alpha^{(\mathbb{R}_+)}\big(f\cdot\chi_{[0,a]}\big)(x) \\ & +\chi_{(a,\infty)}(x) M_\alpha^{(\mathbb{R}_+)}\big(f\cdot\chi_{(a,\infty)}\big)(x) \\ =: \; & M_\alpha^{(1)} f(x) + M_\alpha^{(2)} f(x) + M_\alpha^{(3)} f(x) + M_\alpha^{(4)} f(x). \end{aligned}$$

Since $\|wf\|_{L^{p(\cdot)}(\mathbb{R}_+)} < \infty$ we have that $\|wf\|_{L^{p(\cdot)}([0,a])} < \infty$. Applying now Theorem 1.1 we find that $\|vM_\alpha^{(1)} f\|_{L^{p(\cdot)}(\mathbb{R}_+)} < \infty$. Further, observe that

$$M_\alpha^{(2)} f(x) \le \sup_{h>a-x} \frac{1}{h} \int_a^{x+h} |f(y)|dy \le \big(M_\alpha^{(\mathbb{R}_+)} f\big)(a) < \infty.$$

Hence,

$$\|vM_\alpha^{(2)} f\|_{L^{p(\cdot)}(\mathbb{R}_+)} \le \big(M_\alpha^{(\mathbb{R}_+)} f\big)(a)\cdot \|v\|_{L^{p(\cdot)}([0,a])} < \infty.$$

Let us use the following representation for $M_\alpha^{(3)} f(x)$:

$$\begin{aligned} \big(M_\alpha^{(3)} f\big)(x) &= \chi_{(a,2a]}(x) M_\alpha^{(\mathbb{R}_+)}\big(f\cdot\chi_{[0,a]}\big)(x) + \chi_{(2a,\infty)}(x) M_\alpha^{(\mathbb{R}_+)}\big(f\cdot\chi_{[0,a]}\big)(x). \\ &=: \big(\overline{M}_\alpha^{(3)} f\big)(x) + \big(\widetilde{M}_\alpha^{(3)} f\big)(x). \end{aligned}$$

It is easy to check that for $x \in (a, 2a]$,

$$\big(\overline{M}_\alpha^{(3)} f\big)(x) \le \sup_{h>a-x} \frac{1}{(a-x+h)^{1-\alpha}} \int_{x-h}^{a} |f(y)|dy \le \big(M_\alpha^{(\mathbb{R}_+)} f\big)(a).$$

Consequently,

$$\|v\overline{M}_\alpha^{(3)} f\|_{L^{p(\cdot)}(\mathbb{R}_+)} \le \|f\|_{L^{p_c}\left((a,2a]\right)} \big(M_\alpha^{(\mathbb{R}_+)} f\big)(a) < \infty,$$

because $v^{p(\cdot)}(\cdot)$ is locally integrable on $\mathbb{R}_+$. Further we have that for $x > 2a$,

$$\big(\widetilde{M}_\alpha^{(3)} f\big)(x) \le \frac{1}{(x-a)^{1-\alpha}} \int_0^a |f(y)|dy.$$

Hence, by using Hölder's inequality in $L^{p(\cdot)}$ spaces, we find that

$$\begin{aligned} \left\| v\widetilde{M}_\alpha^{(3)} f \right\|_{L^{p(\cdot)}(\mathbb{R}_+)} &\le \left\| \frac{v(x)}{(x-a)^{1-\alpha}} \right\|_{L^{p_c}\left((2a,\infty)\right)} \left(\int_0^a |f(y)|dy \right) \\ &\le \left\| \frac{v(x)}{(x-a)^{1-\alpha}} \right\|_{L^{p_c}\left((2a,\infty)\right)} \\ & \quad \|fw\|_{L^{p(\cdot)}\left((0,a]\right)} \|w^{-1}\|_{L^{p'(\cdot)}\left((0,a]\right)} \\ &= I_1 \cdot I_2 \cdot I_3. \end{aligned}$$

Since $I_2 < \infty$ and $I_3 < \infty$, we need to show that $I_1 < \infty$. This follows from the fact that condition (1.3) yields

$$\big\|v\overline{M}_\alpha\big(w^{-(p_c)'}\chi_I\big)\big\|_{L^{p_c}\big((2a,\infty)\big)} \le \big\|w^{1-(p_c)'}(\cdot)\chi_I(\cdot)\big\|_{L^{p_c}\big((2a,\infty)\big)}, \quad I \subset (2a,\infty), \tag{1.4}$$

where $\overline{M}_\alpha$ is the maximal operator defined on $(2a,\infty)$ as follows:

$$\big(\overline{M}_\alpha f\big)(x) = \sup_{h>0}\frac{1}{h^{1-\alpha}}\int\limits_{(2a,\infty)\cap(x-h,x+h)}|f(y)|dy.$$

Using the result by E. Sawyer see [31] (see also [13], Ch. 4) for Lebesgue spaces with constant parameter, we see that (1.4) implies the inequality

$$\big\|v\overline{M}_\alpha f\big\|_{L^{p_c}\big((2a,\infty)\big)} \le c\big\|fw\big\|_{L^{p_c}\big((2a,\infty)\big)}.$$

Since

$$\overline{M}_\alpha f(x) \ge \frac{1}{(x-a)^{1-\alpha}}\int\limits_{2a}^{x}|f(y)|dy \quad \text{for} \quad x > 2a,$$

we have that for the Hardy operator

$$\big(H_a f\big)(x) = \int\limits_{2a}^{x} f(t)dt, \qquad x > 2a,$$

the two-weight inequality

$$\big\|v(x)(x-a)^{\alpha-1}H_a f\big\|_{L^{p_c}\big((2a,\infty)\big)} \le \big\|wf\big\|_{L^{p_c}\big((2a,\infty)\big)} \tag{1.5}$$

holds. Let us recall that (see e.g. [25], Section 1.3) necessary condition for (1.5) is that

$$\sup_{t>2a}\bigg(\int\limits_{t}^{\infty}\bigg[\frac{v(x)}{(x-a)^{1-\alpha}}\bigg]^{p_c}dx\bigg)^{\frac{1}{p_c}}\bigg(\int\limits_{2a}^{t}w^{1-(p_c)'}(x)dx\bigg)^{\frac{1}{(p_c)'}} < \infty.$$

Hence,

$$\int\limits_{2a}^{\infty}\bigg[\frac{v(x)}{(x-a)^{1-\alpha}}\bigg]^{p_c}dx = \int\limits_{2a}^{3a}(\cdots) + \int\limits_{3a}^{\infty}(\cdots)$$

$$\le a^{\alpha-1}\int\limits_{2a}^{3a}\big(v(y)\big)^{p_c} + \int\limits_{3a}^{\infty}\bigg[\frac{v(x)}{(x-a)^{1-\alpha}}\bigg]^{p_c}dx < \infty.$$

It remains to estimate $I := \|vM_\alpha^{(4)}f\|_{L^{p(\cdot)}(\mathbb{R}_+)}$. But $I < \infty$ because of the two-weight result by E. Sawyer [31] (see also [13], Ch.4) for the maximal operator defined on (a,∞) in Lebesgue spaces with constant exponent. Sufficiency is proved.

Necessity follows easily by taking the test functions $f(\cdot) = \chi_I(\cdot)w^{-p'(\cdot)}(\cdot)$ in the two–weight inequality. $\square$

The next statement follows in the same way as the previous one; therefore we omit the proof.

Theorem 1.3. *Let $0 \le \alpha < 1$, $1 < p_- \le p \le p_+ < \infty$, and let $p \in LH(\mathbb{R})$. Suppose that there is a positive number a such that $w^{-p'(\cdot)}(\cdot) \in DC([-a,a])$ and $p \equiv p_c \equiv$const outside $[-a,a]$. Then the inequity*

$$\|vM_\alpha^{(\mathbb{R})} f\|_{L^{p(\cdot)}(\mathbb{R})} \le \|wf\|_{L^{p(\cdot)}(\mathbb{R})},$$

holds if and only if there is a positive constant b such that for all bounded intervals $I \subset \mathbb{R}$,

$$\|vM_\alpha^{(\mathbb{R})}(w^{-p'(\cdot)}\chi_I)\|_{L^{p(\cdot)}(\mathbb{R})} \le c\|w^{1-p'(\cdot)}\|_{L^{p(\cdot)}(I)} < \infty.$$

2 Integral operators on $\mathbb{R}_+$

In this section we derive two–weight criteria of other type for the operators

$$(\mathcal{H}f)(x) = (\text{p.v.}) \int_0^\infty \frac{f(t)}{x-t} dt, \quad x \in \mathbb{R}_+,$$

$$(\mathcal{M}f)(x) = \sup_{I \ni x} \frac{1}{|I|} \int_I |f(t)| dt, \quad x \in \mathbb{R}_+,$$

provided that weights are monotonic, where the supremum is taken over all finite intervals $I \subset \mathbb{R}_+$ containing x.

In this section we shall use the notation

$$g_- := g_-(\mathbb{R}_+); \quad g_+ := g_+(\mathbb{R}_+),$$

for a measurable function $g : \mathbb{R}_+ \to \mathbb{R}_+$.

First we present the following statement regarding the weighted Hardy transform

$$(H_{v,w}f)(x) = v(x) \int_0^x f(t)w(t)dt$$

and its dual

$$(H'_{v,w}f)(x) = v(x) \int_x^\infty f(t)w(t)dt$$

defined on $\mathbb{R}_+$.

Theorem A. Let $1 < p_- \le p(x) \le q(x) \le q_- < \infty$ and let $p, q \in LH(\mathbb{R}_+)$. Suppose that $p = p_c \equiv const$, $q = q_c \equiv const$ outside some interval $(0,a)$. Then

(i) the operator $H_{v,w}$ is bounded from $L^{p(\cdot)}(\mathbb{R}_+)$ to $L^{q(\cdot)}(\mathbb{R}_+)$ if and only if

$$D := \sup_{t>0} D(t) := \sup_{t>0} \|v\|_{L^{q(\cdot)}\big((t,\infty)\big)} \|w\|_{L^{p'(\cdot)}\big((0,t)\big)} < \infty;$$

(ii) the operator $H'_{v,w}$ is bounded from $L^{p(\cdot)}(\mathbb{R}_+)$ $L^{q(\cdot)}(\mathbb{R}_+)$ if and only if

$$D' := \sup_{t>0} D'(t) := \sup_{t>0} \|v\|_{L^{q(\cdot)}\big((0,t)\big)} \|w\|_{L^{p'(\cdot)}\big((t,\infty)\big)} < \infty.$$

Proof. We prove part (i). Part (ii) follows from the duality arguments. Let $\|f\|_{L^{q(\cdot)}(\mathbb{R}_+)} \le 1$. We represent $H_{v,w}f$ as follows:

$$H_{v,w}f(x) = \chi_{[0,a]}v(x)\int_0^x f(t)w(t)dt + \chi_{(a,\infty)}v(x)\int_0^x f(t)w(t)dt := H^{(1)}_{v,w}f(x) + H^{(2)}_{v,w}f(x).$$

Observe that the condition $D<\infty$ implies that

$$D^{(a)} := \sup_{0<t<a} \|v\|_{L^{q(\cdot)}\big((t,a)\big)}\|w\|_{L^{p'(\cdot)}\big((0,t)\big)} < \infty.$$

Consequently (see [22]),

$$\|H^{(1)}_{v,w}f\|_{L^{q(\cdot)}(\mathbb{R})} \le c\|f\|_{L^{p(\cdot)}([0,a])} \le c.$$

It remains to estimate $\|H^{(2)}_{v,w}f\|_{L^{q(\cdot)}(\mathbb{R}_+)}$. Let $\|g\|_{L^{q'(\cdot)}(\mathbb{R}_+)} \le 1$. We have that

$$\int_0^\infty (H^{(2)}_{v,w}f)(x)g(x)dx = \int_a^\infty (H^{(2)}_{v,w}f)(x)g(x)dx$$

$$\le \int_a^\infty v(x)\Big(\int_a^x f(t)w(t)dt\Big)g(x)dx + \Big(\int_a^\infty v(x)g(x)dx\Big)\Big(\int_0^a f(t)w(t)dt\Big) := S_1 + S_2.$$

We can now apply the boundedness of the Hardy transform $T^{(a)}_{v,w}f(x) = v(x)\int_a^x f(t)w(t)dt$ from $L^{p_c}([a,\infty))$ to $L^{q_c}([a,\infty))$ (see e.g. [25], Section 1.3) because

$$\sup_{t>a} \|v\|_{L^{q_c}\big((t,\infty)\big)}\|w\|_{L^{(p_c)'}\big((a,t)\big)} \le D < \infty.$$

Consequently, by this fact and Hölder's inequality we derive that

$$S_1 \le \|T^{(a)}_{v,w}f\|_{L^{q_c}([a,\infty))}\|g\|_{L^{q_c}([a,\infty))} \le c\|f\|_{L^{p(\cdot)}(\mathbb{R}_+)} \le C.$$

Applying Hölder's inequality for $L^{p(\cdot)}$ spaces we find that

$$S_2 \le \Big(\int_a^\infty v(x)g(x)dx\Big)\|f\|_{L^{p(\cdot)}([0,a])}\|w\|_{L^{p'(\cdot)}([0,a])} \le C.$$

Necessity follows by the standard way choosing the appropriate test functions. □

Theorem B ([12]). $1<p_- \le p_+ < \infty$. *Suppose that* $p \in LH(\mathbb{R}_+)$ *and that* $p = p_c = const$ *outside some interval. Then the inequality*

$$\|vTf\|_{L^{p(\cdot)}(\mathbb{R}_+)} \le c\|wf\|_{L^{p(\cdot)}(\mathbb{R}_+)}, \tag{2.1}$$

where T *is* $\mathcal{M}$ *or* $\mathcal{H}$, *holds if*

(i) $H_{\overline{v},\widetilde{w}}$ is bounded in $L^{p(\cdot)}(\mathbb{R})$, where $\overline{v}(x) := \frac{v(x)}{x}$, $\widetilde{w}(x) := \frac{1}{w(x)}$;

(ii) $H'_{v,\widetilde{w}_1}$ is bounded in $L^{p(\cdot)}(\mathbb{R})$, where $\widetilde{w}_1(x) := \frac{1}{w(x)x}$;

(iii)

$$v_+([x/4,4x]) \le cw(x) \text{ a.e. or } v(x) \le cw_-([x/4,4x]) \text{ a.e.} \tag{2.2}$$

Theorems A and B imply the following statement:

Theorem 2.1. *Let $1 < p_- \le p_+ < \infty$ and let $p \in LH(\mathbb{R}_+)$. Suppose that $p = p_c \equiv$ const outside some interval $[0,a]$. Suppose also that v and w are weights on $\mathbb{R}_+$. Then the inequality* (2.1)*, where T is $\mathcal{M}$ or $\mathcal{H}$, holds if*

(i)

$$E_1 := \sup_{t>0} E_1(t) := \sup_{t>0} \|v(x)x^{-1}\|_{L^{p(x)}\big((t,\infty)\big)} \|w^{-1}\|_{L^{p'(\cdot)}\big((0,t)\big)} < \infty; \tag{2.3}$$

(ii)

$$E_2 := \sup_{t>0} E_2(t) := \sup_{t>0} \|v\|_{L^{p(\cdot)}\big((0,t)\big)} \|w^{-1}(x)x^{-1}\|_{L^{p'(x)}\big((0,t)\big)} < \infty; \tag{2.4}$$

(iii) condition (2.2) is satisfied.

Now we prove the next statement.

Theorem 2.2. *Let $1 < p_- \le p_+ < \infty$ and let $p \in LH(\mathbb{R}_+)$. Suppose that $p = p_c \equiv$ const outside some interval $[0,a]$. Suppose also that v and w are positive increasing functions on $\mathbb{R}_+$. Then inequality* (2.1)*, where T is $\mathcal{M}$ or $\mathcal{H}$, holds if and only if* (2.3) *is satisfied.*

Proof. Sufficiency. Taking Theorem 2.1 into account it is enough to see that condition (2.3) implies conditions (2.4) and (2.2). For (2.2) we will show that there is a positive constant c such that for all $t > 0$ inequality

$$v(4t) \le cw(t), \quad t > 0. \tag{2.5}$$

holds. Indeed, inequality (1.1) with respect to the Lebesgue measure $d\mu(x) = dx$ and the exponent $r = p'$ which belongs to $LH([0,a])$, for small t, yields that

$$E_1(t) \ge \|\chi_{[t,4t]}(\cdot)| \cdot |^{-1}\|_{L^{p(\cdot)}_{v(\cdot)}(\mathbb{R}_+)} \parallel \chi_{[0,t/4]}(\cdot)w^{-1}(\cdot) \parallel_{L^{p'(\cdot)}(\mathbb{R}_+)}$$

$$\ge c\frac{v(t)}{t} t^{\frac{1}{p_-([t,4t])}} w^{-1}(t/4) t^{\frac{1}{(p')_-([0,t/4])}} \ge c\frac{v(t)}{w(t/4)} t^{-1} t^{\frac{1}{p_-([0,4t])}} t^{\frac{1}{(p')_-([0,t/4])}} = c\frac{v(t)}{w(t/4)}.$$

Further, for large t, we have that

$$E_1(t) \ge \|v(x)x^{-1}\chi_{(t,2t)}(x)\|_{L^{p_c}(\mathbb{R}_+)} \|\chi_{[t/8,t/4]}(\cdot)w^{-1}(\cdot)\|_{L^{p'_c}(\mathbb{R}_+)} \ge c\frac{v(t)}{w(t/4)} t^{-1} t^{\frac{1}{p_c}} t^{\frac{1}{(p_c)'}} = c\frac{v(t)}{w(t/4)}$$

Thus, condition (2.2) is satisfied.

Taking into account the fact that v and w are increasing and inequality (2.5) we can easily conclude that condition (2.4) is satisfied.

Necessity. First observe that inequality (2.1) implies that $\|w^{-1}\|_{L^{p'(\cdot)}(0,t)} < \infty$ for all $t > 0$.

Let $T = \mathcal{M}$. Then using the obvious inequality

$$\mathcal{M}f(x) \ge \frac{c}{x} \int_0^x f(t)dt, \quad x > 0,$$

and taking into account Theorem A we have necessity for $\mathcal{M}$. Let now $T = \mathcal{H}$. We take $f \ge 0$ so that $\|f\|_{L^{p(\cdot)}_w(\mathbb{R}_+)} \le 1$. Then we have that

$$\|v\mathcal{H}f\|_{L^{q(\cdot)}(\mathbb{R}_+)} \le C. \tag{2.6}$$

Obviously, (2.6) yields that

$$C \geq \|v\mathcal{H}f\|_{L^{q(\cdot)}(\mathbb{R}_+)} \geq \|\chi_{(t,\infty)}(\cdot)v\mathcal{H}f\|_{L^{p(\cdot)}(\mathbb{R}_+)}.$$

If f has support on $(0,t)$, $t>0$, then this inequality implies that

$$C \geq \left\|\chi_{(t,\infty)}(\cdot)v(\cdot)\Big(\int\limits_0^t \frac{f(y)}{\cdot - y}dy\Big)\right\|_{L^{p(\cdot)}(\mathbb{R}_+)} \geq c\left\|\chi_{(t,\infty)}(x)v(x)x^{-1}\right\|_{L^{p(\cdot)}(\mathbb{R}_+)}\Big(\int\limits_0^t f(y)dy\Big).$$

By taking now supremum with respect to f and using the inequality

$$\|g\|_{L^{p(\cdot)}} \leq \sup_{\|h\|_{L^{p'(\cdot)}}\leq 1}\left|\int gh\right|,$$

(see e.g. [28]) we have necessity. $\square$.

Acknowledgements

The authors were partially supported by the Georgian National Science Foundation Grant (project numbers: No. GNSF/ST09/23/3-100 and No. GNSF/ST07/3-169).

References

[1] C. Capone, D. Cruz-Uribe SFO and A. Fiorenza, The fractional maximal operator on variable L^p spaces, *Revista Mat. Iberoamericana* **3(23)** (2007), 747–770.

[2] D. Cruz-Uribe SFO, A. Fiorenza, J. M. Martell and C. Perez, The boundedness of classical operators on variable L^p spaces., *Ann. Acad. Sci. Fenn. Math.* **31** (2006), 239–264.

[3] D. Cruz-Uribe, SFO, A. Fiorenza and C.J. Neugebauer, The maximal function on variable L^p spaces, *Ann. Acad. Sci. Fenn. Math.* **28** (2003), 223-238, and 29 (2004), 247–249.

[4] L. Diening, Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$, *Math. Inequal. Appl.* **7(2)** (2004), 245–253.

[5] L. Diening, Riesz potentials and Sobolev embeddings on generalized Lebesgue and Sobolev spaces $L^{p(\cdot)}$ and $W^{k,p(\cdot)}$., *Math. Nachr.* **268** (2004), 31–34.

[6] L. Diening and P. Hästö, Muckenhoupt weights in variable exponent spaces., *Preprint, Available at* http://www.helsinki.fi/~pharjule/varsob/publications.shtml .

[7] L. Diening and M. Ružička, Calderón-Zygmund operators on generalized Lebesgue spaces $L^{p(\cdot)}$ and problems related to fluid dynamics., *J. Reine Angew. Math.* **63** (2003), 197–220.

[8] L. Diening and S. Samko, Hardy inequality in variable exponent Lebesgue spaces, *Frac. Calc. Appl. Anal.* **10** (2007), 1–18.

[9] D. E. Edmunds, V. Kokilashvili, and A. Meskhi, Bounded and compact integral operators., *Mathematics and Its Applications, Kluwer Academic Publishers, Dordrecht, Boston, London,* 2002.

[10] D. E. Edmunds, V. Kokilashvili, and A. Meskhi, A trace inequality for generalized potentials in Lebesgue spaces with variable exponent., *J. Funct. Spaces Appl.* **2** (2004), 55–69.

[11] D. E. Edmunds, V. Kokilashvili, and A. Meskhi, On the boundedness and compactness of the weighted Hardy operators in $L^{p(x)}$ spaces. *Georgian Math. J.* **12** (2005), 27–44.

[12] D. E. Edmunds, V. Kokilashvili, and A. Meskhi, Two-weight estimates in $L^{p(x)}$ spaces with applications to Fourier series., *Houston J. Math.* **2** (2009), 665–689.

[13] J. García–Cuerva and J. L. Rubio de Francia, Weighted norm inequalities and related topics, *North-Holland Mathematics Studies*, 116, *Mathematical Notes*, 104, *North-Holland Publishing Co., Amsterdam*, 1985.

[14] P. Harjulehto, P. Hästö, and M. Pere, Variable exponent Lebesgue spaces on metric spaces: the Hardy-Littlewood maximal operator. *Real Anal. Exchange* **30** (2004-2005), 87–104.

[15] V. Kokilashvili and M. Krbec, Weighted inequalities in Lorentz and Orlicz spaces. *Singapore, New Jersey, London, Hong Kong, World Scientific,* 1991.

[16] V. Kokilashvili and A. Meskhi, Weighted criteria for generalized fractional maximal functions and potentials in Lebesgue spaces with variable exponent, *Integr. Trans. Spec. Func.* **18** (2007), 609–628.

[17] V. Kokilashvili and A. Meskhi, Maximal and singular integrals in Morrey spaces with variable exponent. *Arm. J. Math.* **1** (2008), 18–28.

[18] V. Kokilashvili and S. Samko, Maximal and fractional operators in weighted $L^{p(x)}$ spaces, *Rev. Mat. Iberoamericana* **20** (2004), 493–515.

[19] V. Kokilashvili and S. Samko, On Sobolev theorem for Riesz-type potentials in Lebesgue spaces with variable exponent, *Z. Anal. Anwendungen* 22 (2003), 899–910.

[20] V. Kokilashvili and S. Samko, The maximal operator in weighted variable spaces on metric spaces, *Proc. A. Razmadze Math. Inst.* **144** (2007), 137–144.

[21] V. Kokilashvili and S. Samko, Operators of harmonis analysis in weighted spaces with non–standard growth, *J. Math. Anal. Appl.* **352(1)** (2009), 15–34. (2008).

[22] T. S. Kopaliani, On some structural properties of Banach function spaces and boundedness of certain integral operators. *Czechoslovak Math. J.* **54(129)** (2004), 791–805.

[23] O. Kovácik and J. Rákosník, On spaces $L^{p(x)}$ and $W^{k,p(x)}$, *Czechoslovak Math. J.* **41(4)** (1991), 592–618.

[24] F.Mamedov and Y. Zeren, On a two–weighted estimation of maximal operator in the Lebesgue space with variable exponent. *Annali di Matematica*, Doi 10.1007/s10231-010-0149.

[25] V. G. Maz'ya, *Sobolev spaces*, Springer-Verlag, Berlin-Heidelberg, 1984.

[26] A. Nekvinda, Hardy-Littlewood maximal operator on $L^{p(\cdot)}(R^n)$, *Math. Ineq. Appl.* **7** (2004), No. 2, 255–265.

[27] N.G. Samko, S.G. Samko, and B. Vakulov, Weighted Sobolev theorem in Lebesgue spaces with variable exponent, *J. Math. Anal. Appl.* **335** (2007), p. 560-583.

[28] S. Samko, Convolution type operators in $L^{p(x)}$, *Integral Transf. and Spec. Funct.* **7(1–2)** (1998), 123–144.

[29] S. Samko, Convolution and potential type operators in $L^{p(x)}(\mathbb{R}^n)$, *Integr. Transf. and Spec. Funct.* **7(3–4)** (1998), 261–284.

[30] S. Samko and B. Vakulov, Weighted Sobolev theorem with variable exponent, *J. Math. Anal. Appl.* **310** (2005), 229–246.

[31] E. Sawyer, Two–weight norm inequalities for certain maximal and integral operators, in: Harmonic analysis, Minneapolis, Minn. 1981, pp. 102–127, Lecture Notes in Math. Vol. 908, Springer, Berlin, New York, 1982.

[32] E. T. Sawyer and R. L. Wheeden, Carleson conditions for the Poisson integral, *Indiana Univ. Math. J.* **40** (1991), No. 2, 639–676.

[33] I. I. Sharapudinov, On a topology of the space $L^{p(t)}([0,1])$. *Mat. Zametki* **26** (1979), 613–632.

Authors' Addresses:

V. Kokilashvili: A. Razmadze Mathematical Institute, 1. M. Aleksidze Str., 0193 Tbilisi, Georgia and Faculty of Exact and Natural Sciences, I. Javakhishvili Tbilisi State University 2, University St., Tbilisi 0143 Georgia
e-mail: kokil@rmi.acnet.ge.

A. Meskhi: A. Razmadze Mathematical Institute, 1. M. Aleksidze Str., 0193 Tbilisi, Georgia and Department of Mathematics, Faculty of Informatics and Control Systems, Georgian Technical University, 77, Kostava St., Tbilisi, Georgia.
e-mail: meskhi@rmi.acnet.ge